MOP WEEK 1 EXERCISE

a. Prove that an increasing function, $f:\mathbb{R}\to\mathbb{R}$, is of bounded variation over $[a,b]$, $a,b\in\mathbb{R}$, $a\le b$.

b. Prove that if $g:\mathbb{R}\to\mathbb{R}$ is the difference of increasing functions, g_1, g_2 say, then g is of bounded variation over $[a,b]$.

c. Give an example of a function, defined on $[0,1]$, which is not of bounded variation. Hint: it doesn't have to be continuous on $[0,1]$.

d. Recall the definition of quadratic variation of an L^2-martingale, (X_t):

$$\langle X\rangle_t = \lim_{\theta} \sum_{\theta} \Delta X_{t_i}^2$$

for partitions θ of $[0,t]$. Assume that the limit exists in probability. Show that,

(i) If $0\le s\le t$ then $0\le \langle X\rangle_s \le \langle X\rangle_t$

(ii) For $t\ge 0$, $\langle X\rangle_t$ is $\mathcal{F}_t$-measurable.

(iii) For a simple process, $f(s)=\sum_i f_{t_{i-1}} I_{(t_{i-1},t_i]}(s)$, where $f_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$ measurable, prove the isometry property:

$$\left\| \int_0^t f(s)\,dX_s \right\|_2^2 = E\left(\int_0^t |f_s|^2 d\langle X\rangle_s \right).$$

State your assumptions about the simple process f.